1901001101050001
EXAMINATION FEBRUARY-MARCH 2024
MASTER OF ARTS PART-I (EXTERNAL)
MATHEMATICS
GRAPH THEORY & DISCRETE STRUCTURE - LEVEL 5

[**Time: As Per Schedule]** **[Max. Marks:100]**

Instructions:
1. Fill up strictly the following details on your answer book
   a. Name of the Examination: **MASTER OF ARTS PART – I (EXTERNAL)**
   b. Name of the Subject: **MATHEMATICS GRAPH THEORY & DISCRETE STRUCTURE – LEVEL 5**
   c. Subject Code No: **1901001101050001**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. Each question carries equal marks.
6. Follow usual notations and conventions.
7. Use of non-programmable scientific calculator is permitted.

Seat No:

Student's Signature

Q.1 (a) (i) Prove that a simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges. **7**

(ii) Prove that in a connected graph G with exactly $2k$ odd vertices, there exist k edge disjoint subgraphs such that they together contain all edges of G and that each is a Unicursal graph.

(b) Prove that an Euler graph G is arbitrarily traceable from vertex v in G if and only if every circuit in G contains v. **7**

(c) Discuss the Utilities Problem. **6**

OR

(a) Prove that a given connected graph G is an Euler graph if and only if all vertices of G are of even degree. **7**

(b) Define: Path and Circuit. **7**
Prove that if a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

(c) (i) Prove that vertex connectivity ≤ edge connectivity ≤ $d\ (v_i)$, where v_i is the vertex with the smallest degree. **6**

(ii) Prove that in a connected graph G, the complement of a cut-set in G does not contain a spanning tree and the complement of a spanning tree does not contain a cut-set.

Q.2 (a) Define: Spanning tree, Branch and Chord. **7**
Prove that any given edge of a connected graph G is a branch of some spanning tree. Is it also true that any arbitrary edge of G is a chord for some spanning tree of G?

(b) (i) What is the maximum possible height of an n-vertex binary tree? **7**

(ii) Prove that every connected graph has at least one spanning tree.

(c) Prove that any connected graph with n vertices and $n - 1$ edges is a tree. **6**

OR

(a) Prove that the ring sum of any two cut-sets in a graph is either a third cut-set or an edge disjoint union of cut-sets. **7**

(b) Define: Tree, Rooted tree and Binary tree. **7**
Prove that a graph G with n vertices, $n - 1$ edges and no circuits is connected.

(c) Prove that a tree with n vertices have $n - 1$ edges. **6**

Q.3 (a) (i) Prove that concatenation is not commutative except when V contains a single element. **7**

(ii) Prove that, no element of V^* is invertible, except the empty string.

(b) (i) Prove that any subgraph g of a connected graph G is contained in some spanning tree of G if and only if g contains no circuits. **7**

(ii) Prove that a pendant edge in a connected graph G is contained in every spanning tree.

(c) Let f from $< X, \circ >$ onto $< Y, \oplus >$ be a homomorphism. Then prove that E_f is a congruence relation on $< X, o >$ given by $x_1 E_f x_2 \Leftrightarrow f(x_1) = f(x_2)$, for any $x_1, x_2 \in X$. **6**

OR

(a) If $A(G)$ is an incidence matrix of a connected graph G with n vertices, prove that the rank of $A(G)$ is $n - 1$. **7**

(b) Discuss the observation about the incidence matrix. **7**

(c) Define: Semigroup and Monoid. **6**
Let X be a non-empty set then prove that $< X^X, \circ >$ is a monoid, where $(f \circ g)(x) = f(g(x))$ for $f, g \in X^X$ and $x \in X$.

Q.4 (a) State and prove the Fermat's theorem. **7**

(b) Prove that the multiplicative inverse of and element $a \in Z_m$ exists and is unique iff GCD$(a, m) = 1$ and $a \neq 0$. **7**

(c) For the set of natural numbers N, prove that $< N, + >$ is a semigroup. Is the set of odd non-negative integers form a subsemigroup for $< N, + >$? Justify. **6**

OR

(a) (i) Prove that for a semigroup homomorphism idempotent element is preserved. **7**

(ii) Prove that for a semigroup homomorphism commutativity is preserved.

(b) (i) What are the advantages of decimal and binary number system? **7**

(ii) What are the disadvantages of residue number system?

(c) Define: Monoid and Submonoid. **6**
Prove that the set of idempotent elements of a commutative monoid forms a submonoid.

Q.5 (a) Prove that for any mapping from a Boolean algebra which preserves the operation $*$ and $\oplus$, the image set is also a Boolean algebra. **7**

(b) Prove that direct product of two lattices is again a lattice. **7**

(c) (i) Prove that the set of elements which have complements form a sublattice in a bounded distributive lattice. **6**

(ii) Show that any atom which is $\leq a$ must appear in the join representation of the element a.

OR

(a) State and prove the cancellation law of arithmetic. **7**

(b) Discuss the procedure to find x in a mixed based number system. **7**

(c) Prove that every chain is a distributive lattice. **6**

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